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A Singularity Theory Approach to a Semilinear Boundary Value Problem

K. J. BROWN

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom

AND

J. E. FURTER*

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

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1. INTRODUCTION

In this paper we discuss various aspects of obtaining bifurcation diagrams in the (u, λ)-plane for the nonlinear boundary value problem

$$\begin{aligned} -u_{tt} &= \lambda u(1-u)(u-a) \qquad \text{for} \quad t \in (0, 1) \\ u(0) &= b = u(1), \end{aligned} \tag{1.1}$$

where a, b, λ are real parameters with $0 < a, b < 1$. This problem arises in population genetics with the unknown function u corresponding to the relative frequency of a gene. The parameter a is given by a combination of the death rates of the various genotypes involved, b gives the frequency of the gene on the boundary of the region, and λ is the reciprocal of the diffusion rate of the gene (see Aronson and Weinberger [1] for further details). Solutions of (1.1) correspond to possible steady states of the gene populations. We are interested mainly in how the bifurcation diagram changes as a and b vary and we investigate this question by exploiting singularity theory for bifurcation problems as developed by Golubitsky and Schaeffer [5].

Equation (1.1) is a simple example of a semilinear elliptic boundary value problem and a standard method of investigating the bifurcation of

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solutions of such equations is by using the Lyapunov–Schmidt technique in which the existence of solutions for λ close to an eigenvalue of the corresponding linear problem is reduced to the study of a single scalar equation. In Section 2 we suppose that for fixed values of a, b, and λ, the linearisation of (1.1) with respect to a solution $\bar{u}$ is non-invertible and using the Lyapunov–Schmidt technique we show that the existence of solutions close to $(\bar{u}, \bar{\lambda})$ is equivalent to finding the solutions of a single equation $g(x, \lambda) = 0$ where x and λ are real variables.

In Section 3 we study the special case where $a = b$, $\bar{u} \equiv a$, and $\lambda = \lambda_k$ is the kth eigenvalue of the linearisation of (1.1) about $\bar{u}$. It is shown that the bifurcation diagram of $g(x, \lambda) = 0$ is a pitchfork if k is even or $a = 1/2$ and a transcritical bifurcation otherwise. In Section 4 we investigate whether a and b are unfolding parameters for the bifurcation diagrams of Section 3, i.e., whether it is possible to obtain all possible bifurcation diagrams close to the pitchfork or transcritical bifurcation by varying a and b in a neighbourhood of their original values. We show that a and b are unfolding parameters if k is odd whereas if k is even the bifurcation diagram remains a pitchfork for all a and b.

Our results imply that many steady-state solutions exist for values of the parameters close to the pitchfork bifurcation point. The results also make clear how bifurcation diagrams obtained by Eilbeck in [4] in a numerical study of (1.1) are related to each other and indicate parameter values for which there exist more steady-state solutions than found in the cases considered in [4].

Another approach to Eq. (1.1) is to consider the phase plane associated with the differential equation $-u_{tt} = u(1-u)(u-a)$. Solutions of (1.1) correspond to trajectories in the phase plane which join the line $u = b$ to itself in time $\sqrt{\lambda}$. If $T(\rho)$ denotes the time taken for a trajectory passing through the point (b, ρ) to return to the line $u = b$ (at the point $(b, -\rho)$), the number of solutions of (1.1) is determined by the properties of the function T, which is often referred to as the time map. The time map is given by a certain integral and properties of T can be deduced by using elementary calculus. The time map has been studied extensively by other authors, in particular we shall refer later to the results of Smoller and Wasserman [6, 7]. In Section 5 we discuss some connections between the results of the earlier sections and the time map and show how it is possible to use the phase plane approach to obtain information on how the local bifurcation diagrams previously obtained join up in a global bifurcation diagram.

Our approach can greatly help to organize the results obtained by the time map technique. Problem (1.1) is actually equivalent to the more general cubic problem studied by Smoller and Wasserman in [7]. They studied $-u_{tt} = -\mu(u-\alpha)(u-\beta)(u-\gamma)$ for $t \in [0, 1]$, $\alpha < \beta < \gamma$, and Dirichlet boundary conditions. This is equivalent to (1.1) under the change in coordinates $b=\alpha/(\gamma-\alpha)$ and $a=(\beta-\alpha)/(\gamma-\alpha)$. This change of variable prevents a straightforward comparison of our results and those in [7]. Nevertheless our results make it easier to understand some of the phenomena obtained in [7], e.g., the so-called "higher order bifurcation" discussed in the concluding remarks of [7] where the view is taken that the bifurcation diagrams shown in Figs. 11 and 12 of [7] demonstrate the fact that bifurcation diagrams may change discontinuously with parameters. If the singularity theory viewpoint is adopted, however, the dependence of the diagrams on the parameters becomes more transparent. If we consider the family of all solutions instead of simply positive solutions, then Fig. 11 also contains a parabola-like branch of negative solutions and Fig. 12 becomes a transcritical bifurcation. Thus the lack of continuity disappears as we discuss in Theorem 3.1 how Fig. 11 arises in the unfolding of Fig. 12. The analysis in Section 3, however, is valid only locally whereas the results in [7] are global in nature.

Finally in Section 6 we show that any critical point of the time map $T(\rho)$ corresponds to an x-critical point of the bifurcation function $g(x,\lambda)$, of the same order, i.e., $T'(\rho_0)=\cdots=T^{(k-1)}(\rho_0)=0$, $T^{(k)}(\rho_0)\neq 0$ if and only if $g_x(x_0,\lambda)=\cdots=g_{x^{k-1}}(x_0,\lambda)=0$, $g_{x^k}(x_0,\lambda)\neq 0$. This generalizes the results of Brunosky and Chow [2] who discuss the non-critical case where $T'(p_0)\neq 0$.

Our approach seems capable of generalisation to the case where the non-linearity is a polynomial of degree k and the $(k-1)$ unfolding parameters are taken as $(k-2)$ roots of the polynomial and the value of the solution on the boundary. An analysis similar to that described in Section 3 would be possible around the special point where each of the $(k-2)$ roots is equal to 1/2.

Many of our results would also hold in higher space dimensions (where u_{xx} is replaced by Δu) with the generic assumption that the eigenspace of the linearized problem is one dimensional. In the one dimensional case study of the phase plane suggests that the results which can be established locally by singularity theory in fact hold for a wide range of parameter values; in the case of higher space dimensions results of Budd [3] suggest that the local analysis will be valid in a much more limited range.

2. Lyapunov–Schmidt Technique

In order to apply the Lyapunov–Schmidt technique it is more convenient to work with zero Dirichlet boundary conditions. Thus we use the change of variable $v=u-b$ to transform (1.1) into

$$-v_{tt}=\lambda h(v,a,b) \quad \text{for} \quad 0<t<1; \qquad v(0)=0=v(1), \tag{2.1}$$

where $h(v,a,b)=-v^3+(1+a-3b)v^2+(2ab-3b^2+2b-a)v+b(b-a)(1-b)$.

It is also convenient to rescale the control parameters as

$$a = \tfrac{1}{2} + \tfrac{1}{2}\varepsilon_1; \qquad b = \tfrac{1}{2} + \tfrac{1}{2}\varepsilon_1 + \varepsilon_2 \tag{2.2}$$

so that we may rewrite (2.1) as

$$-v_{tt} = \lambda h(v, \varepsilon_1, \varepsilon_2) \quad \text{for} \quad 0 < t < 1; \qquad v(0) = 0 = v(1) \tag{2.3}$$

where $h(v, \varepsilon_1, \varepsilon_2) = -v^3 - (\varepsilon_1 + 3\varepsilon_2)v^2 + (1/4 - (1/4)\varepsilon_1^2 - 2\varepsilon_1\varepsilon_2 - 3\varepsilon_2^2)v + (\varepsilon_2/4)(1 - (\varepsilon_1 + 2\varepsilon_2)^2)$.

To apply the Lyapunov–Schmidt technique to (2.3) we must introduce an appropriate function space setting. Let

$$V = \{u \in C^2[0, 1]: u(0) = 0 = u(1)\}.$$

Then we can express (2.3) as an operator equation

$$F(v, \lambda, \varepsilon_1, \varepsilon_2) = v_{tt} + \lambda h(v, \varepsilon_1, \varepsilon_2) = 0, \tag{2.4}$$

where $F: V \times R^3 \to C[0, 1]$ is a smooth function.

Suppose that (2.4) has a solution $(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$. We shall discuss the solution set of (2.4) in a neighbourhood of $(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$. Let $L = F_v(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, the Frechet derivative of F with respect to v. Thus $L\varphi = \varphi_{tt} + \bar{\lambda} h_v(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)\varphi$. If L is invertible, the implicit function theorem shows that there is a unique branch of solutions $(v(\lambda), \lambda, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$ passing through $(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$. On the other hand, if L is not invertible, the null-space of L consists of the non-zero solutions of the linearisation of (2.3), viz.,

$$-v_{tt} = \bar{\lambda} h_v(\bar{v}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)v; \qquad v(0) = 0 = v(1) \tag{2.5}$$

and Sturm–Liouville theory shows that the null space is one dimensional, being generated by v_0 say. By projecting onto the subspace generated by v_0, we next show that (2.4) is equivalent to a single scalar equation—the bifurcation equation.

Let $\langle , \rangle$ and $\| \; \|$ denote the usual inner product and associated norm on $L^2(0, 1)$. Let $X = \text{span}\{v_0\}$ and let $Y = \{u \in C[0, 1]: \langle u, v_0 \rangle = 0\}$. Then $\langle Lu, v \rangle = \langle u, Lv \rangle$ for all $u, v \in V$, $X =$ null space of L, and $L: Y \cap V \to Y$ is an isomorphism. Let P denote the projection of $C[0, 1]$ into X, i.e., $Pv = (\langle v, v_0 \rangle / \langle v_0, v_0 \rangle) v_0$ and let $Q = I - P$. Any function $v \in V$ can be written uniquely as $v = xv_0 + w$ where $x \in R$ and $w \in Y \cap V$. Thus Eq. (2.4) is equivalent to the pair of equations

$$PF(\bar{v}+xv_0+w, \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2)=0 \tag{2.6}$$

$$QF(\bar{v}+xv_0+w, \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2)=0. \tag{2.7}$$

Since $L: Y \cap V \to Y$ is invertible, it follows that QL; i.e., the Frechet derivative of $w \to QF(\bar{v}+w, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$ at $w=0$ is invertible. Hence by the implicit function theorem there exists a unique smooth function $w(x, \lambda, \varepsilon_1, \varepsilon_2)$ mapping into Y such that

$$QF(\bar{v}+xv_0+w(x, \lambda, \varepsilon_1, \varepsilon_2), \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2)=0 \tag{2.8}$$

for all $(x, \lambda, \varepsilon_1, \varepsilon_2)$ in a neighbourhood of $(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$. Hence (2.4) is equivalent to a single equation

$$PF(\bar{v}+xv_0+w(x, \lambda, \varepsilon_1, \varepsilon_2), \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2)=0$$

which is in turn equivalent to the scalar equation

$$G(x, \lambda, \varepsilon_1, \varepsilon_2)=\langle F(\bar{v}+xv_0+w(x, \lambda, \varepsilon_1, \varepsilon_2), \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2), v_0\rangle=0. \tag{2.9}$$

The solutions and so the bifurcation diagram of (2.4) in the (λ, v)-plane when $\varepsilon_1=\bar{\varepsilon}_1$ and $\varepsilon_2=\bar{\varepsilon}_2$ are determined by the equation

$$g(x, \lambda)=G(x, \lambda, \bar{\varepsilon}_1, \bar{\varepsilon}_2)=0. \tag{2.10}$$

The function g is termed the bifurcation function of (2.4) associated with the solution $(\bar{v}, \bar{\lambda}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$.

Finally in this section we derive some simple properties of the functions w and g. It is easy to see that $w(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)=0$ and so $g(0, 0)=0$. We now calculate $w_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$. Since $w(x, \lambda, \varepsilon_1, \varepsilon_2) \in Y$ for all appropriate $(x, \lambda, \varepsilon_1, \varepsilon_2)$ it follows that all derivatives of w with respect to these variables must be in Y. In particular $w_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2) \in Y$. Differentiating (2.8) with respect to x gives

$$QF_u(\bar{v}+xv_0+w(x, \lambda, \varepsilon_1, \varepsilon_2), \bar{\lambda}+\lambda, \varepsilon_1, \varepsilon_2)\,[v_0+w_x(x, \lambda, \varepsilon_1, \varepsilon_2)]=0 \tag{2.11}$$

and so, setting $(x, \lambda, \varepsilon_1, \varepsilon_2)=(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, $QL(v_0+w_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2))=0$. Since $QLv_0=0$, we must have that $QLw_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)=0$. But $QL: V \cap Y \to Y$ is an isomorphism and so $w_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)=0$.

Differentiating (2.9) with respect to x and with respect to λ gives respectively

$$g_x(0, 0)=\langle L(v_0+w_x(0, 0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)), v_0\rangle=\langle Lv_0, v_0\rangle=0 \tag{2.12}$$

and

$$g_\lambda(0,0)=\langle w_\lambda(0,0,\bar{\varepsilon}_1,\bar{\varepsilon}_2), Lv_0\rangle+\langle h(\bar{v},\bar{\varepsilon}_1,\bar{\varepsilon}_2), v_0\rangle$$
$$=\int_0^1 h(\bar{v}(t),\bar{\varepsilon}_1,\bar{\varepsilon}_2)\, v_0(t)\,dt. \tag{2.13}$$

3. Local Analysis of the Bifurcation Equation

It is easy to see that when $a=b$ (i.e., when $\varepsilon_2=0$) Eqs. (2.1) and (2.3) have the trivial solution $v\equiv 0$ for all values of λ. We study bifurcation points for Eq. (2.3) on this trivial branch. Since $h_v(0,\bar{\varepsilon}_1,0)=(1/4)(1-\varepsilon_1^{-2})$, Eq. (2.5) becomes

$$-v_{tt}=\tfrac{1}{4}\lambda(1-\varepsilon_1^{-2})v; \qquad v(0)=0=v(1)$$

and so bifurcation occurs at $\lambda=\lambda_k=4(k\pi)^2(1-\varepsilon_1^{-2})$ for $k=1,2,\ldots$.

We shall use the singularity theory techniques developed by Golubitsky and Schaeffer in [5] to study the bifurcation function corresponding to these bifurcation points and so obtain a good understanding of the local bifurcation diagrams. We recall some of the basic definitions of [5].

Let E be the ring of germs of smooth functions $f\colon R^2\to R$. We denote by $\mathcal{M}$ the maximal ideal of E containing all functions f such that $f(0,0)=0$ and by $\mathcal{M}^k$ the ideal $\{f\in E\colon D^\alpha f(0)=0 \text{ for } |\alpha|\leqslant k-1\}$.

Let $f,g\in E$. Then f and g are contact equivalent if there are smooth changes in coordinates $T(x,\lambda)$, $X(x,\lambda)$, $\Lambda(\lambda)$ such that

$$f(x,\lambda)=T(x,\lambda)\,g(X(x,\lambda),\Lambda(\lambda))$$

with $T(0,0)\neq 0$, $X(0,0)=0$, $X_x(0,0)>0$, $\Lambda(0)=0$, and $\Lambda_\lambda(0)>0$. Thus, if f and g are contact equivalent, the equation $g(x,\lambda)=0$ can be transformed into the equation $f(x,\lambda)=0$ by a smooth change of coordinates and the bifurcation diagrams associated with the two equations are qualitatively similar. Using singularity theory we can find a normal form for the bifurcation function g in (2.10), i.e., a simpler expression, in fact a polynomial, to which g is contact equivalent. Then the bifurcation diagram of the normal form and so qualitatively the bifurcation diagram for g can be easily determined.

THEOREM 3.1. (i) *Suppose* $\bar{\varepsilon}_2=0$ *and* $\bar{\varepsilon}_1\neq 0$. *Then the bifurcation function at* $(\bar{v},\bar{\lambda},\bar{\varepsilon}_1,\bar{\varepsilon}_2)=(0,\lambda_k,\bar{\varepsilon}_1,0)$ *is contact equivalent to a pitchfork with normal form* $-x^3+\lambda x$ *if* k *is even and to a transcritical bifurcation with normal form* $-x^2+\lambda x$ *if* k *is odd.*

(ii) *Suppose* $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$ (*i.e.*, $a = b = 1/2$). *Then the bifurcation function at* $(0, \lambda_k, 0, 0)$ *is contact equivalent to a pitchfork.*

Proof. Let $g(x, \lambda)$ denote the bifurcation function associated with the solution $(v, \lambda, \bar{\varepsilon}_1, \bar{\varepsilon}_2) = (0, \lambda_k, \bar{\varepsilon}_1, 0)$. Since $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ are regarded as fixed parameters while considering g, we suppress the dependence of functions on ε_1 and ε_2 as far as possible, writing $F(v, \lambda, \bar{\varepsilon}_1, 0)$ as $F(v, \lambda)$, etc.

Since

$$Lv = F_u(0, \lambda_k)v = v_{tt} + \lambda_k h_v(0)v = v_{tt} + k^2\pi^2 v$$

we may choose $v_0 = \sin k\pi t$.

Finding the normal form (or in the terminology of [4] solving the recognition problem) of the bifurcation function involves checking various conditions on g and its derivatives at $(x, \lambda) = (0, 0)$. Since all derivatives of g are evaluated at $(0, 0)$ only, we write $g = g(0, 0)$, $g_x = g_x(0, 0)$, etc. It was shown in Section 2 that $g = g_x = 0$ and by (2.13)

$$g_\lambda = \int_0^1 h(0, \bar{\varepsilon}_1, 0) \sin k\pi t \, dt = 0.$$

It is shown in [5] that provided $g = g_x = g_\lambda = 0$ then

(i) g is contact equivalent to a transcritical bifurcation if and only if $g_{xx} \neq 0$, $g_{\lambda x}^2 - g_{xx} g_{\lambda\lambda} > 0$.

(ii) g is contact equivalent to a supercritical pitchfork if and only if $g_{xx} = 0$ and $g_{xxx} g_{\lambda x} < 0$.

We showed in Section 2 that $w_x(0, 0) = 0$. We now compute $w_\lambda(0, 0)$ and $w_{xx}(0, 0)$. Again since all derivatives are evaluated at $(0, 0)$ we write $w_{xx}(0, 0)$ as w_{xx}, etc. Differentiating (2.8) with respect to λ gives

$$QLw_\lambda + QF_\lambda(0, \lambda_k) = 0.$$

Since $F_\lambda(0, \lambda_k) = h(0, \bar{\varepsilon}_1, 0) = 0$, $QLw_\lambda = 0$. Hence as $QL\colon V \cap Y \to Y$ is an isomorphism, $w_\lambda = 0$.

It follows from differentiating (2.11) again with respect to x that w_{xx} is the unique solution in Y of the second order O.D.E.

$$\phi'' + k^2\pi^2\phi = 2\bar{\varepsilon}_1 \lambda_k Q v_0^2. \tag{3.1}$$

We now compute the appropriate derivatives of g. Differentiating (2.9) twice with respect to x gives

$$\begin{aligned} g_{xx} &= \langle F_{uu}(0, \lambda_k)(v_0 + w_x)^2 + Lw_{xx}, v_0 \rangle \\ &= -2\bar{\varepsilon}_1 \lambda_k \int_0^1 \sin^3 k\pi t \, dt. \end{aligned} \tag{3.2}$$

If $\bar{\varepsilon}_1 \neq 0$ and k is odd, $g_{xx} = -(8/3k\pi)\lambda_k \bar{\varepsilon}_1 \neq 0$ and so g is contact equivalent to a transcritical bifurcation provided $g_{\lambda x}^2 - g_{xx} g_{\lambda\lambda} > 0$. Differentiating (2.9) with respect to x and then λ gives

$$g_{x\lambda} = \langle F_{u\lambda}(0, \lambda_k) v_0, v_0 \rangle = \int_0^1 h_v(0, \bar{\varepsilon}_1, 0) v_0^2 \, dt = \frac{1}{8}(1 - \bar{\varepsilon}_1^2).$$

Differentiating (2.9) twice with respect to λ gives

$$g_{\lambda\lambda} = \langle F_{uu}(0, \lambda_k) w_\lambda^2 + 2F_{u\lambda}(0, \lambda_k) w_\lambda + Lw_{\lambda\lambda} + F_{\lambda\lambda}(0, \lambda_k), v_0 \rangle = 0.$$

Thus $g_{x\lambda}^2 - g_{xx} g_{\lambda\lambda} > 0$ and so g is contact equivalent to a transcritical bifurcation when k is odd and $\bar{\varepsilon}_1 \neq 0$.

If k is even or $\bar{\varepsilon}_1 = 0$ then $g_{xx} = 0$ and we must compute g_{xxx}. Differentiating (2.9) three times with respect to x gives

$$g_{xxx} = \langle F_{uuu}(0, \lambda_k) v_0^3 + 3F_{uu}(0, \lambda_k) v_0 w_{xx} + Lw_{xxx}, v_0 \rangle$$

and so

$$g_{xxx} = -6\lambda_k \int_0^1 [v_0^4 + \bar{\varepsilon}_1 v_0^2 w_{xx}] \, dt. \tag{3.3}$$

Suppose $\bar{\varepsilon}_1 = 0$. Then it follows from (3.1) that $w_{xx} = 0$ and so $g_{xxx} = -6\lambda_k \int_0^1 v_0^4 \, dt = -(9/4)\lambda_k < 0$. Suppose now that k is even. Then $v_0^2 = \sin^2 k\pi x$ and $\langle v_0^2, v_0 \rangle = 0$. Thus $Qv_0^2 = v_0^2$ and so w_{xx} is the unique solution in Y of

$$\phi'' + k^2\pi^2\phi = 2\bar{\varepsilon}_1 \lambda_k \sin^2 k\pi t.$$

Hence

$$w_{xx}(t) = \frac{\bar{\varepsilon}_1 \lambda_k}{k^2\pi^2}\left[1 + \frac{1}{3}\cos 2k\pi t - \frac{4}{3}\cos k\pi t\right].$$

Therefore by (3.3)

$$g_{xxx} = -6\lambda_k \int_0^1 [\sin^4 k\pi t + \bar{\varepsilon}_1 \sin^2 k\pi t \, w_{xx}(t)] \, dt = -\frac{9}{4}\lambda_k - \frac{5}{2}\left(\frac{\bar{\varepsilon}_1 \lambda_k}{k\pi}\right)^2 < 0.$$

Therefore, if $\bar{\varepsilon}_1 = 0$ or k is even, $g_{xx} = 0$ but $g_{xxx} g_{x\lambda} < 0$ and so g is contact equivalent to the pitchfork. This completes the proof.

4. Universal Unfoldings

In this section we investigate whether ε_1 and ε_2 (and so a and b) are unfolding parameters for the bifurcation diagrams obtained in the previous section, i.e., whether it is possible to obtain all bifurcation diagrams close to the pitchfork or transcritical bifurcation by varying $(\varepsilon_1, \varepsilon_2)$ in a neighbourhood of their values at the bifurcation point.

First we recall the relevant material from [5]. Let E_{2+k} be the ring of smooth maps $F: R^{2+k} \to R$ such that $F(0,0,0)=0$. If $f \in E$ and $F \in E_{2+k}$ such that $F(x, \lambda, 0) = f(x, \lambda)$, we say that F is an unfolding with k parameters of f. If $F \in E_{2+k}$ and $G \in E_{2+l}$ are unfoldings of $f \in E$, we say that F reduces into G if there exists T, X, and Λ which are unfoldings of the identity in the relevant spaces and a germ $\Psi: R^k \to R^l$ with $\psi(0)=0$ such that

$$F(x, \lambda, \alpha) = T(x, \lambda, \alpha)\, G(X(x, \lambda, \alpha), \Lambda(\lambda, \beta), \psi(\alpha)).$$

If $f \in E$, an unfolding F of f is called versal if every other unfolding of f reduces into F. The smallest number k of parameters needed for an unfolding F of f (i.e., $F \in E_{2+k}$) to be versal is termed the codimension of f; the corresponding unfoldings are called universal. It can be proved that all universal unfoldings of f are contact equivalent. The universal unfolding characterizes all possible bifurcation diagrams of small perturbations of f; i.e., every possible bifurcation diagram lying close to the bifurcation diagram of f is exhibited by $F(x, \lambda, \alpha) = 0$ for some small fixed values of the parameters α. It is proved in [5] that the unfolding parameter space, i.e., α space, can be split up into a finite number of regions, all germs $F(x, \lambda, \alpha)$ corresponding to each region being contact equivalent and so possessing qualitatively the same bifurcation diagram in (x, λ)-space. These regions are separated by the following transition varieties

$$B = \{\alpha: \text{there exist } x, \lambda \text{ such that } F(x, \lambda, \alpha) = F_x(x, \lambda, \alpha) = F_\lambda(x, \lambda, \alpha) = 0\}$$

$$H = \{\alpha: \text{there exist } x, \lambda \text{ such that } F(x, \lambda, \alpha) = F_x(x, \lambda, \alpha) = F_{xx}(x, \lambda, \alpha) = 0\}$$

$$DL = \{\alpha: \text{there exist } x, y, \lambda \text{ such that } F(x, \lambda, \alpha) = F(y, \lambda, \alpha) = F_x(x, \lambda, \alpha) = F_x(y, \lambda, \alpha) = 0\}.$$

If $\alpha \in B$, H, or DL, the bifurcation diagrams of $F(x, \lambda, \alpha) = 0$ contain respectively at least a bifurcation point, a hysteresis point, and two limit points at the same value of λ, respectively.

For example, the pitchfork bifurcation $f(x, \lambda) = -x^3 + \lambda x$ is of codimension 2 with universal unfolding $-x^3 + \lambda x + \alpha - \beta x^2$ where α and β are the unfolding parameters. In this case the transition varieties are $B=\{(\alpha, \beta): \alpha=0\}$, $H=\{(\alpha, \beta): \alpha=-\beta^3/27\}$, and DL is empty.

We now investigate when ε_1 and ε_2 are unfolding parameters of the bifurcation function g of the previous section.

THEOREM 4.1. *Suppose $\bar{\varepsilon}_1=\bar{\varepsilon}_2=0$ and k is odd so that the bifurcation function $g(x, \lambda)$ has a pitchfork singularity. Then the function $G(x, \lambda, \varepsilon_1, \varepsilon_2)$ is a universal unfolding of $g(x, \lambda)$ with unfolding parameters ε_1 and ε_2.*

Proof. We work with germs around $(x, \lambda)=(0,0)$ or $(x, \lambda, \varepsilon_1, \varepsilon_2)=(0,0,0,0)$ and so unless otherwise stated all derivatives are evaluated at these points. It is shown in [5] that $G(x, \lambda, \varepsilon_1, \varepsilon_2)$ is a universal unfolding of the pitchfork singularity $g(x, \lambda)$ if

$$\det \begin{vmatrix} g_x & g_{xx} & g_{\lambda x} & g_{xxx} \\ g_\lambda & g_{\lambda x} & g_{\lambda\lambda} & g_{xx\lambda} \\ G_{\varepsilon_1} & G_{x\varepsilon_1} & G_{\lambda\varepsilon_1} & G_{xx\varepsilon_1} \\ G_{\varepsilon_2} & G_{x\varepsilon_2} & G_{\lambda\varepsilon_2} & G_{xx\varepsilon_2} \end{vmatrix} \neq 0. \tag{4.1}$$

We have already shown that $g_x=g_\lambda=g_{xx}=g_{\lambda\lambda}=0$, that $g_{\lambda x}=1/8$, and that $g_{xxx}=-(9/4)\,\lambda_k$. We now compute a sufficient number of the remaining terms in the above determinant to enable us to calculate its value.

First we must compute some more derivatives of w. We have already shown that $w_{xx}=0$ in this case. Differentiating (2.8) with respect to ε_1 gives

$$QLw_{\varepsilon_1}+QF_{\varepsilon_1}(0, \lambda_k, 0, 0)=0.$$

But $F_{\varepsilon_1}(0, \lambda_k, 0, 0)=\lambda_k h_{\varepsilon_1}(0,0,0)=0$. Hence $QLw_{\varepsilon_1}=0$ and so, since $QL\colon V\cap Y\to Y$ is an isomorphism, $w_{\varepsilon_1}=0$.

Similarly by differentiating (2.11) with respect to ε_1, it can be shown that $w_{x\varepsilon_1}=0$.

By carrying out the appropriate differentiations on (2.9), it can be shown that $G_{\varepsilon_1}=G_{x\varepsilon_1}=G_{\lambda\varepsilon_1}=0$ and that

$$G_{xx\varepsilon_1}=\lambda_k h_{vv\varepsilon_1}(0,0,0)\int_0^1 v_0^3\,dt=-\frac{8k\pi}{3}$$

and

$$G_{\varepsilon_2}=\frac{1}{4}\lambda_\kappa\int_0^1 v_0\,dt=\frac{k\pi}{2}.$$

Thus the determinant in $(4.1)=-k^2\pi^2/48$ and so the proof is complete.

The universal unfolding $H(X, \Lambda, \alpha, \beta)=-X^3+\Lambda X+\alpha-\beta X^2$ of the normal form of the pitchfork $-X^3+\Lambda X$ is shown in Fig. 4.1. It follows from [5] that there is a contact equivalence between $G(x, \lambda, \varepsilon_1, \varepsilon_2)$ and $H(X, \Lambda, \alpha, \beta)$; i.e., there exist smooth maps T, Λ, and ϕ such that

$$G(x, \lambda, \varepsilon_1, \varepsilon_2) = T(x, \lambda, \varepsilon_1, \varepsilon_2)\, H(X, \lambda, \varepsilon_1, \varepsilon_2), \Lambda(\lambda, \varepsilon_1, \varepsilon_2), \phi(\varepsilon_1, \varepsilon_2)).$$

It is straightforward to check that these mappings must be of the form

$$\begin{aligned} T(x, \lambda, \varepsilon_1, \varepsilon_2) &= 1 + \mathcal{M} \\ X(x, \lambda, \varepsilon_1, \varepsilon_2) &= a_k x + \mathcal{M}^2 \\ \Lambda(\lambda, \varepsilon_1, \varepsilon_2) &= b_k \lambda + \mathcal{M}^2 \\ \phi(\varepsilon_1, \varepsilon_2) &= (c_k \varepsilon_2 + \mathcal{M}^2, d_k \varepsilon_1 + \mathcal{M}^2), \end{aligned}$$

where a_k, b_k, c_k, and d_k are given by

$$a_k^3 = \frac{3}{8} k^2 \pi^2; \qquad b_k = (8a_k)^{-1}; \qquad c_k = \frac{k\pi}{2}; \qquad d_k = a_k^{-2} \frac{4k\pi}{3}.$$

Because of the above explicit form of the transformation between G and H, it can be seen that G can be represented as an unfolding of g as shown in Fig. 4.2. The fact that the orientation of branches is preserved between the two diagrams depends on the special form of the linear part of the transformation $\Phi: (x, \lambda) \to (X, \Lambda)$. The definition of contact equivalence ensures that the matrix corresponding to this transformation is upper triangular with positive diagonal entries but this in itself does not imply

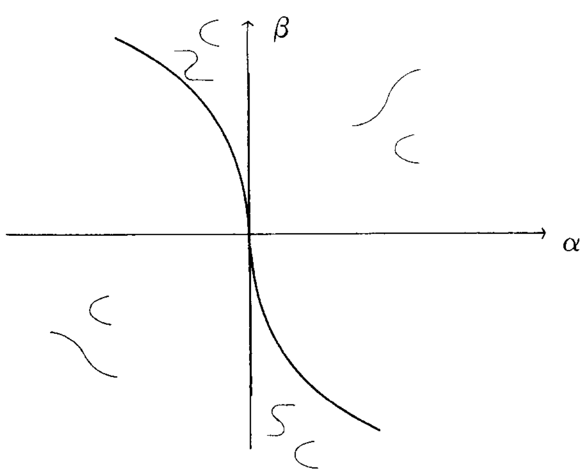

FIG. 4.1. The unfolding in $\alpha - \beta$ space.

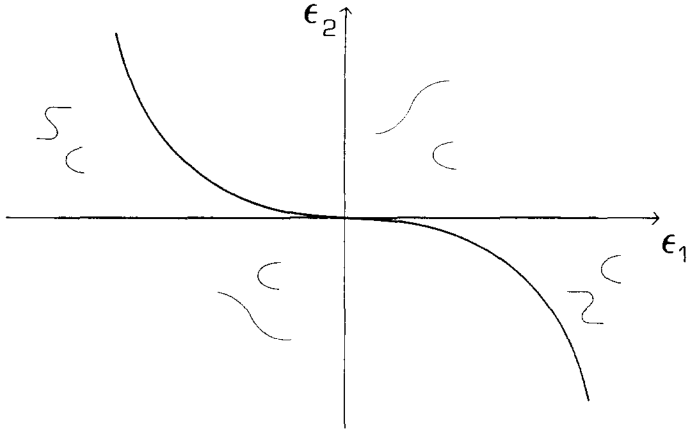

FIG. 4.2. The unfolding in $\varepsilon_1 - \varepsilon_2$ space.

that orientation is preserved, e.g., $x^2 - \lambda x$ and $X^2 + \Lambda X$ are contact equivalent with transformation $X: x - \lambda$, $\Lambda = \lambda$. In this case, however, the matrix corresponding to the linear part of Φ is diagonal and so orientation is preserved. This is important as the sign of x determines the sign of the derivative and so the general nature of the corresponding solution of the differential equation. The transition varieties in $(\varepsilon_1, \varepsilon_2)$-space are

$$B = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = 0\} \qquad \text{and}$$

$$H = \left\{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = -\frac{1}{k^2\pi^2} 2^{13} 3^{-8} \varepsilon_1^3 + \mathcal{M}^4\right\}.$$

Thus the transition varieties remain qualitatively the same for all odd k but H approaches B as $k \to \infty$.

As $(\varepsilon_1, \varepsilon_2) \to (a, b)$ is a simple affine map it is easy to see that the unfolding diagram in (a, b) space is as shown in Fig. 4.3.

By using arguments similar to those above the following result can be obtained for the case $\bar{\varepsilon}_1 \neq 0$, $\bar{\varepsilon}_2 = 0$.

THEOREM 4.2. *Suppose k is odd, $\bar{\varepsilon}_1 \neq 0$, $\bar{\varepsilon}_2 = 0$ so that the bifurcation function $g(x, \lambda)$ has a transcritical bifurcation. Then the function $G(\cdot, \cdot, \bar{\varepsilon}_1, \cdot)$ is a universal unfolding of g with unfolding parameter ε_2.*

In this case it can be checked that G reduces to $-X^2 + \Lambda X + \gamma$ the

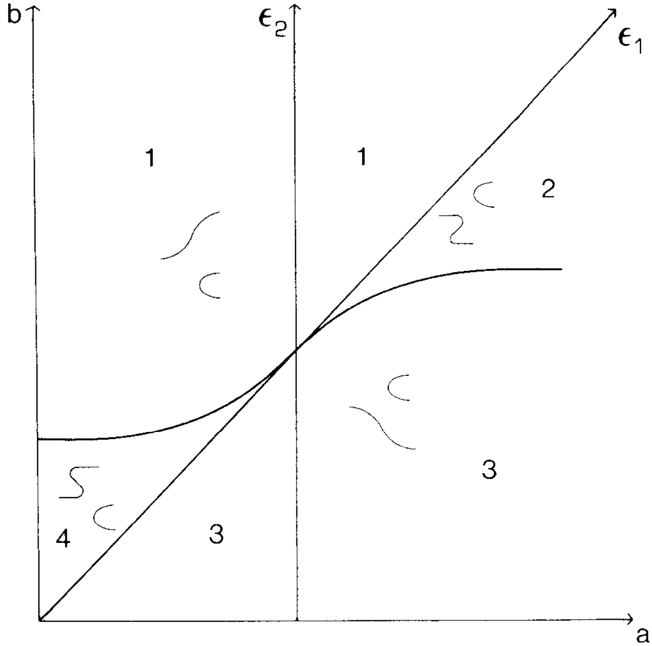

FIG. 4.3. The unfolding in $a-b$ space.

canonical unfolding for the transcritical bifurcation using the following change of variable

$$T = 1 + \mathscr{M}$$
$$X = a_k x + \mathscr{M}^2$$
$$\Lambda = b_k \lambda + \rho_k(\varepsilon_1 - \bar{\varepsilon}_1) + \mathscr{M}^2$$
$$\phi = c_k \varepsilon_2 + \mathscr{M}^2,$$

where

$$a_k > 0, \qquad a_k^2 = \frac{16k\pi\bar{\varepsilon}_1}{3(1-\varepsilon_1^{-2})}, \qquad b_k = \frac{1-\varepsilon_1^{-2}}{8a_k},$$

$$c_k = \frac{2k\pi}{1-\varepsilon_1^{-2}}, \qquad \rho_k = \frac{-(k\pi)^2\bar{\varepsilon}_1}{a_k(1-\bar{\varepsilon}_1^2)}.$$

Theorem 4.1 does not apply to the case where k is even. In this case $G_{\varepsilon_2} = (1/4)\,\lambda_k \int_0^1 v_0\,dt = 0$ and so the determinant in (4.1) equals zero. We now show by using symmetry arguments that ε_1 and ε_2 are not unfolding parameters for the pitchfork in this case.

The O.D.E. (1.2) is invariant with respect to the map $\gamma: C^0[0, 1] \to C^0[0, 1]$ such that $\gamma: u(t) \to u(1-t)$. Along with the identity, γ induces a group action of the group $\{\gamma, I\} \approx \mathbb{Z}_2$ into $C^0[0, 1]$. Since the hypotheses for the standard theorems in equivariant bifurcation theory are satisfied, the bifurcation equation is equivariant for the induced action of Γ into $\ker L$. If k is even the induced action is non-trivial since $\gamma(\sin 2\pi nt) = \sin 2\pi n(1-t) = -\sin 2\pi nt$ and so g is $\mathbb{Z}_2$-invariant. The pitchfork is stable under $\mathbb{Z}_2$ perturbations (see Golubitsky and Schaeffer [5]) and it is easy to see that the perturbations induced by ε_1 and ε_2 possess $\mathbb{Z}_2$ symmetry. Therefore $G(x, \lambda, \varepsilon_1, \varepsilon_2)$ is always contact equivalent to a pitchfork for all small ε_1 and ε_2 and so ε_1 and ε_2 cannot act as unfolding parameters.

If k is odd, the induced action is trivial as $\gamma(\sin(2n+1)\pi t) = \sin(2n+1)\pi t$. Hence the bifurcation function is not symmetric and so ε_1 and ε_2 can and (as is shown in Theorem 4.1) do act as unfolding parameters.

5. Bifurcation Diagrams and the Time Map

The bifurcation diagram associated with (1.1) can also be determined from the phase plane associated with the differential equation. The change of variable $t \to \sqrt{\lambda}\, t$ transforms (1.1) into

$$-u_{tt} = f(u) \text{ for } t \in (0, \sqrt{\lambda}); \qquad u(0) = b = u(\sqrt{\lambda}), \tag{5.1$_\lambda$}$$

where $f(u) = u(u-a)(1-u)$.

The phase plane associated with $-u_{tt} = f(u)$ is shown in Fig. 5.1. There are three equilibrium points $(0, 0)$, $(a, 0)$, and $(1, 0)$. The point $(a, 0)$ is a centre and $(0, 0)$ and $(1, 0)$ are saddle points. There is a homoclinic orbit through $(0, 0)$; inside this homoclinic orbit lie closed trajectories surrounding $(a, 0)$. Solutions of $(5.1)_\lambda$ correspond to trajectories in the phase plane which join the line $u = b$ to itself in time $\sqrt{\lambda}$. Consider trajectory $A_1A_2A_3$ in Fig. 5.1 where A_1 has coordinates $(0, \rho)$. This trajectory corresponds to a solution of $(5.1)_\lambda$ like that shown in Fig. 5.2, the associated value of λ being the square of the time required to traverse the trajectory. This time can be expressed as a function of ρ using elliptic integrals, see Smoller and Wasserman [7]. The function $\rho \to T(\rho)$ is termed the time map associated with the equation. The multiplicity of solutions of $(5.1)_\lambda$ may be deduced from the graph of T, e.g., if $T(\rho_1) = T(\rho_2)$ $(=\gamma$ say) then $(5.1)_\gamma$ has two solutions u_1 and u_2 with $u_1'(0) = \rho_1$, $u_2'(0) = \rho_2$ where $\lambda = \gamma^2$. Other types of solutions may be studied by using other types of trajectories and their

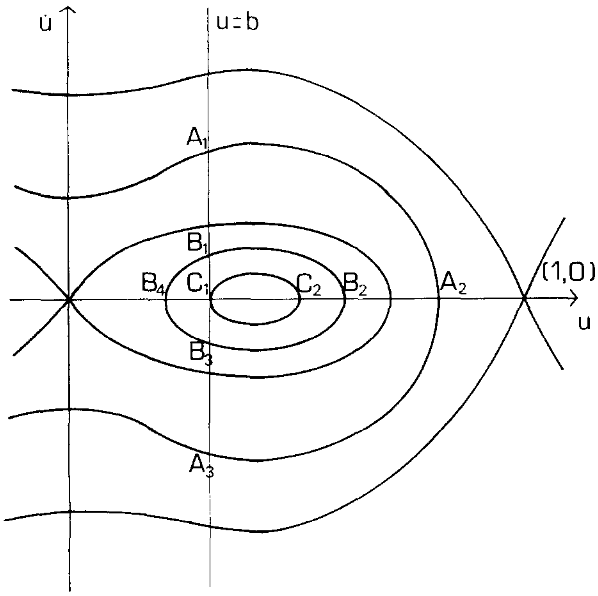

FIG. 5.1. The phase plane for the differential equation.

associated time maps. For example, trajectories $B_3B_4B_1$ and $B_1B_2B_3B_4B_1$ correspond to solutions like those shown in Fig. 5.3.

We now discuss the bifurcation diagrams in the unfolding of G shown in Fig. 4.3. Suppose $b<a$. We consider first solutions of the type shown in Fig. 5.2, i.e., with $u(t)>b$ for all $t \in (0, \sqrt{\lambda})$. Such solutions correspond to trajectories ranging from trajectories such as $A_1A_2A_3$ lying close to the stable and unstable manifolds of the equilibrium point $(1, 0)$ to the small closed trajectory $C_1C_2C_1$ which touches the line $u=b$. As the trajectory $A_1A_2A_3$ approaches the critical manifolds of $(1, 0)$, the time taken to get from A_1 to A_3 and hence the associated value of λ approaches $+\infty$. It can be shown that the time taken to traverse a small closed trajectory surrounding $(a, 0)$ once approaches $2\pi/f'(0)$ as the size of the trajectory tends to 0.

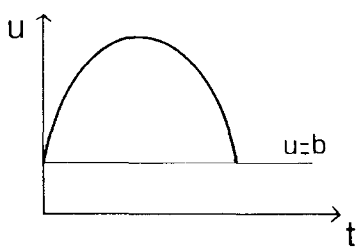

FIG. 5.2. $B_1B_2B_3$ solution.

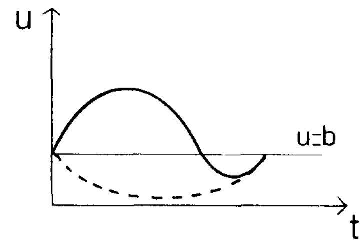

FIG. 5.3. $B_1B_2B_3B_4B_1$, $B_3B_4B_1$ solutions.

Thus when $b-a$ is small the time taken to traverse trajectory $C_1C_2C_1$ is approximately $2\pi/f'(0)$ and the time taken to traverse trajectory $B_1B_2B_3$, which may be regarded roughly speaking as one half of a small closed trajectory, is approximately $\pi/f'(0)$. Thus as we move down the u'-axis from A_1 close to the stable manifold of $(1, 0)$ to C_1 on the u-axis, trajectories starting on the u'-axis correspond to solutions of $(5.1)_\lambda$ where λ decreases from large positive values to approximately $(\pi/f'(0))^2$ and then increases again to approximately $(2\pi/f'(0))^2$. These solutions correspond to the upper branch of solutions possessing a single limit point in regions 2 and 3 in Fig. 4.3. Our analysis shows that close to $a=b=1/2$ the time map described above has a single turning point close to $\lambda=(\pi/f'(0))^2$.

The lower branch of solutions of the bifurcation pictures in regions 2 and 3 correspond to trajectories like $B_3B_4B_1$. It is clear that for such solutions $u(t)<b$ for all t and the corresponding value of λ increases from 0 to ∞ as B_3 moves down the u'-axis from C_1 to close to the homoclinic orbit associated with $(0, 0)$. In region 3, λ increases monotonically but in region 2, λ has two turning points.

We next investigate the bifurcation diagram associated with $\lambda=\lambda_2$. When $a=b$ small closed trajectories correspond to solutions like those shown in Fig. 5.4 for values of λ close to $4\pi^2/[f'(0)]^2$; the value of λ increases as the size of the trajectory increases. It is easy to see that if $u(t)$ is a solution to $(5.1)_\lambda$ then so is $u(1-t)$ so that there is a one–one corre-

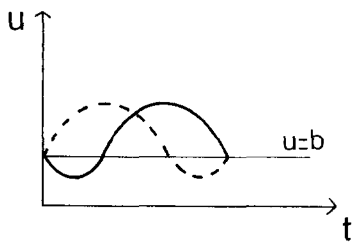

FIG. 5.4. $B_1B_2B_3B_4B_1$, $B_3B_4B_1B_2B_3$ solutions.

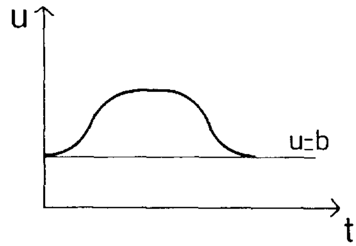

FIG. 5.5. $C_1C_2C_1$ solution.

spondence between the two types of solutions shown in Fig. 5.4. Thus there is a pitchfork bifurcation in this case. If $a \neq 1/2$ and $b \neq a$ the pitchfork bifurcation persists and corresponds to the closed trajectory $C_1C_2C_1$. Suppose that this trajectory is a solution of $(5.1)_{\lambda^*}$. Then (provided $b-a$ is small) $\lambda^* \approx [2\pi/f'(0)]^2$ and this solution also satisfies zero Neumann boundary conditions, see Fig. 5.5. For λ just less than λ^*, we obtain a solution like that shown in Fig. 5.2 associated with a trajectory of the form $B_1B_2B_3$; for λ just greater than λ^* we obtain three solutions of the forms shown in Fig. 5.6 associated with trajectories like $B_1B_2B_3B_4B_1$, $B_3B_4B_1B_2B_3$, and $B_3B_4B_1B_2B_3B_4B_1$, respectively. It is easy to see that each of the four kinds of solutions described above can be regarded as a small perturbation of the solution of $(5.1)_{\lambda^*}$ satisfying zero Neumann boundary conditions. Solutions corresponding to closed trajectories like $B_1B_2B_3B_4B_1$ and $B_3B_4B_1B_2B_3$ can be obtained from one another using the transformation $u(t) \to u(1-t)$. Thus there is a supercritical pitchfork at $\lambda = \lambda^*$.

We can now discuss how the global bifurcation picture is built up from the local bifurcation pictures. Consider the local picture in region 3 in Fig. 4.3 corresponding to λ close to λ_1. The lower curve extends from $\lambda = 0$ to $\lambda = \infty$. The upper branch of the upper curve extends to $\lambda = \infty$; the lower branch of the upper curve however extends to $\lambda = \lambda^* \approx [2\pi/f'(0)]^2$ at which point the pitchfork bifurcation described in the previous paragraph

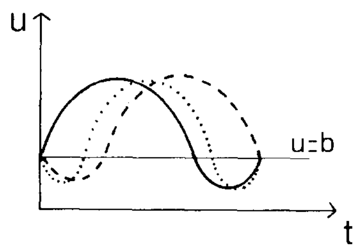

FIG. 5.6. $B_1B_2B_3B_4B_1$, $B_3B_4B_1B_2B_3$, $B_3B_4B_1B_2B_3B_4B_1$ solutions.

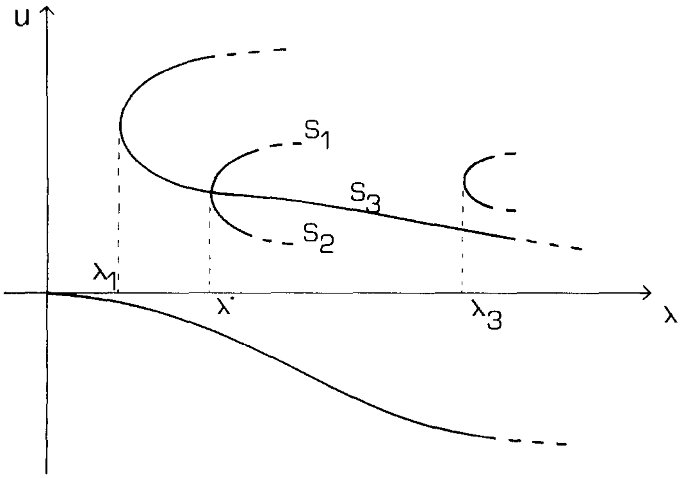

FIG. 5.7. Global bifurcation diagram.

occurs (see Fig. 5.7). It is easy to see from the phase plane that each of the branches S_1, S_2, S_3 extend to $\lambda = \infty$; the branch S_3 forms the lower branch in region 3 of the unfolding associated with $\lambda = \lambda_3$, this local picture also containing an upper branch with a limit point (see Fig. 5.7).

Of course our results are valid only locally, i.e., close to the point $a = b = 1/2$. It is unclear from our analysis what happens away from this point. However, it is easy to see that if the homoclinic orbit associated with $(0, 0)$ cuts the u-axis at $(\hat{b}, 0)$ and $b > \hat{b}$ then only solutions with $u(t) > \hat{b}$ are possible and so the bifurcation diagram contains only a single branch of solutions.

6. Comparison of Critical Points

In this section we investigate the connection between the shape of the solution curves of the bifurcation equation and the graph of the time map $T(\rho)$ described at the start of Section 5. As the number of solutions predicted by the two approaches must be the same it is clear that a turning point or saddle point in the bifurcation diagram must correspond to a stationary point of the same type occurring in the graph of T. It is possible, a priori, however that the two types of curve while predicting the same number of solutions may have different shapes in the neighbourhood of a stationary point, e.g., $T(\rho) \sim (\rho - \rho_0)^4$ and $g(x, \lambda) = x^2 - \lambda$. We show that simple turning points and saddle points for T correspond to the same type of points in the bifurcation diagram $g(x, \lambda) = 0$. We confine our attention to solutions such that $u(t) > b$ for $t \in (0, 1)$; similar considerations apply to solutions of other types.

Let $u(\cdot, \rho)$ denote the solution of the initial value problem

$$-\phi'' = f(\phi) \text{ for } t > 0; \qquad \phi(0) = 0; \qquad \phi'(0) = \rho.$$

Suppose that $T(\bar{\rho}) = \sqrt{\bar{\lambda}}$ and let $v(t, \rho) = u(\sqrt{\bar{\lambda}}\, t, \rho)$. Then $v(\cdot, \rho)$ satisfies

$$-v'' = \bar{\lambda} f(v(t, \rho)) \text{ for } t > 0; \qquad v(0) = 0; \qquad v'(0) = \sqrt{\bar{\lambda}}\, \rho, \tag{6.1}$$

where $'$ denotes differentiation with respect to t.

Differentiating (6.1) with respect to ρ gives

$$-v''_\rho = \bar{\lambda} f'(v(t, \rho))\, v_\rho \text{ for } t > 0, \qquad v_\rho(0) = 0; \qquad v'_\rho(0) = \sqrt{\bar{\lambda}}$$

i.e.,

$$-v''_\rho = \bar{\lambda}\, h'(v)\, v_\rho \text{ for } t > 0, \qquad v_\rho(0) = 0; \qquad v'_\rho(0) = \sqrt{\bar{\lambda}}. \tag{6.2}$$

Comparing (6.2) with (2.5), it can be seen that (2.5) has a non-trivial solution if and only if $v_\rho(1) = 0$.

Since $u(T(\rho), \rho) = 0$, we have

$$v(T(\rho)/\sqrt{\bar{\lambda}}, \rho) = 0. \tag{6.3}$$

Suppose that T has a critical point at $\bar{\rho}$, i.e., $(dT/d\rho)(\bar{\rho}) = 0$ and let $\bar{v}(t) = v(t, \bar{\rho})$. Thus $(\bar{v}, \bar{\lambda})$ is a solution of (2.1).

Differentiating (6.3) with respect to ρ gives

$$(1/\sqrt{\bar{\lambda}})\, v_t(1, \bar{\rho}) \frac{dT}{d\rho}(\bar{\rho}) + v_\rho(1, \bar{\rho}) = 0 \tag{6.4}$$

and so $v_\rho(1, \bar{\rho}) = 0$. Hence $v_0 = v_\rho(\cdot, \bar{\rho})$ is a non-trivial solution of (2.5) and so in the terminology of Section 2 the kernel of $L = F_u(\bar{v}, \bar{\lambda})$ is generated by v_0. Thus using the notation of Section 2 we have that

$$g(x, \lambda) = \langle F(\bar{v} + x v_0 + w(x, \lambda), \bar{\lambda} + \lambda), v_0 \rangle.$$

We showed in Section 2 that $g_x(0, 0) = 0$ and

$$g_\lambda(0, 0) = \int_0^1 h(\bar{v}(t))\, v_0(t)\, dt.$$

Hence

$$g_\lambda(0, 0) = \frac{1}{\sqrt{\bar{\lambda}}} \int_0^{\sqrt{\bar{\lambda}}} h(u)\, u_\rho\, dt.$$

Now, considering the equation of a trajectory in the phase plane,

$$\tfrac{1}{2}[u_t(t,\rho)]^2 + H(u(t,\rho)) = \tfrac{1}{2}\rho^2 \qquad \text{for} \quad t>0,$$

where $H(u)=\int_0^u h(v)\,dv$. Differentiating with respect to ρ we obtain

$$u_t(t,\bar\rho)\,u_{t\rho}(t,\bar\rho) + h(u(t,\bar\rho))\,u_\rho(t,\bar\rho) = \bar\rho.$$

Hence

$$\begin{aligned}\int_0^{\sqrt{\bar\lambda}} h(u)\,u_\rho\,dt &= \bar\rho\sqrt{\bar\lambda} - \int_0^{\sqrt{\bar\lambda}} u_t\,u_{t\rho}\,dt\\ &= \bar\rho\sqrt{\bar\lambda} - u_t u_\rho(\sqrt{\bar\lambda}) + u_t u_\rho(0) + \int_0^{\sqrt{\bar\lambda}} u_{tt}u_\rho\,dt\\ &= \bar\rho\sqrt{\bar\lambda} - \int_0^{\sqrt{\bar\lambda}} h(u)\,u_\rho\,dt.\end{aligned}$$

Hence $2\int_0^{\sqrt{\bar\lambda}} h(u)\,u_\rho\,dt = \bar\rho\sqrt{\bar\lambda}$ and so $g_\lambda(0,0)=\bar\rho/2\neq 0$.

Thus, if $(dT/d\rho)(\bar\rho)=0$, $g_x(0,0)=0$ and $g_\lambda(0,0)\neq 0$ and so g must be contact equivalent to a singularity of the form $x^k \pm \lambda$ where $(dg/dx)(0,0)=\cdots=(d^{k-1}g/dx^{k-1})(0,0)=0$ and $(d^k g/dx^k)(0,0)\neq 0$.

We now investigate the relationship between $(d^2T/d\rho^2)(\bar\rho)$ and $g_{xx}(0,0)$. We have shown that

$$g_{xx}(0,0) = \langle F_{uu}(\bar v,\bar\lambda)\,v_0^2 + F_u(\bar v,\bar\lambda)\,w_{xx}, v_0\rangle = \bar\lambda\int_0^1 h''(\bar v)\,v_0^3\,dt.$$

Differentiating (6.1) twice with respect to ρ gives

$$-v''_{\rho\rho} - \bar\lambda h'(\bar v)\,v_{\rho\rho} = \bar\lambda h''(\bar v)(v_\rho)^2; \qquad v_{\rho\rho}(0)=0.$$

Hence

$$\begin{aligned}\bar\lambda\int_0^1 h''(\bar v)\,v_0^3\,dt &= \int_0^1 [-v''_{\rho\rho} - \bar\lambda h'(\bar v)\,v_{\rho\rho}]\,v_0\,dt\\ &= v_{\rho\rho}(1,\bar\rho)\,v_0'(1).\end{aligned}$$

Thus

$$g_{xx}(0,0) = v_{\rho\rho}(1,\bar\rho)\,v_0'(1).$$

Differentiating (6.3) twice with respect to ρ, gives

$$\frac{1}{\bar\lambda}v_{tt}\left(\frac{dT}{d\rho}\right)^2 + \frac{1}{\sqrt{\bar\lambda}}v_{t\rho}\frac{dT}{d\rho} + \frac{1}{\sqrt{\bar\lambda}}v_t\frac{d^2T}{d\rho^2} + v_{\rho\rho} = 0$$

and so $v_{\rho\rho}(1,\bar\rho)=0$ if and only if $(d^2T/d\rho^2)(\bar\rho)=0$.

Thus we have shown that $(d^2T/d\rho^2)(\bar{\rho})=0$ if and only if $g_{xx}(0,0)=0$. Now suppose that $(dT/d\rho)(\bar{\rho})=(d^2T/d\rho^2)(\bar{\rho})=0$ and so $g_x(0,0)=g_{xx}(0,0)=0$. Then as we have shown above

$$\int_0^1 h''(\bar{v})\, v_0^3\, dt=0. \tag{6.5}$$

Differentiating (2.9) with respect to x gives

$$\begin{aligned} g_{xxx}(0,0) &= \langle F_{uuu}v_0^3+3F_{uu}v_0w_{xx}+F_uw_{xxx}, v_0\rangle \\ &= \bar{\lambda}\int_0^1 h^{(3)}(\bar{v})\, v_0^3\, dt+3\bar{\lambda}\int_0^1 h''(\bar{v})\, v_0^2 w_{xx}\, dt. \end{aligned} \tag{6.6}$$

Now arguments similar to those used in Section 3 show that w_{xx} satisfies the equation

$$Lw_{xx}=QF_{uu}(\bar{v},\bar{\lambda})\, v_0^2; \qquad w_{xx}(0)=0=w_{xx}(1)$$

i.e., $Lw_{xx}=\bar{\lambda}\, Q[h''(\bar{v})\, v_0^2]$. But Eq. (6.5) shows that $h''(\bar{v})\, v_0^2$ lies in the orthogonal complement of span$\{v_0\}$ and so $Q[h''(\bar{v})\, v_0^2]=h''(\bar{v})\, v_0^2$. Hence both w_{xx} and $v_{\rho\rho}$ satisfy the differential equation

$$L\phi=\bar{\lambda}h''(\bar{v})\, v_0^2; \qquad \phi(0)=0=\phi(1)$$

and so w_{xx} and $v_{\rho\rho}$ can differ only by a multiple of v_0. Hence by (6.5)

$$\int_0^1 h''(\bar{v})\, v_0^2\, w_{xx}\, dt=\int_0^1 h''(\bar{v})v_0^2\, v_{\rho\rho}\, dt$$

and so

$$g_{xxx}(0,0)=\bar{\lambda}\int_0^1 h^{(3)}(\bar{v})\, v_0^3+3h''(\bar{v})\, v_0^2v_{\rho\rho}\, dt. \tag{6.7}$$

Differentiating (6.1) three times with respect to ρ gives

$$-v''_{\rho\rho\rho}-\bar{\lambda}h'(\bar{v})\, v_{\rho\rho\rho}=\bar{\lambda}h^{(3)}(\bar{v})\, v_0^3+3\bar{\lambda}\, h''(\bar{v})\, v_{\rho\rho}v_0.$$

Hence

$$\begin{aligned} \bar{\lambda}\int_0^1 [h^{(3)}(\bar{v})\, v_0^3+3h''(\bar{v})\, v_0^2v_{\rho\rho}]\, dt &= \int_0^1 (-v''_{\rho\rho\rho}-\bar{\lambda}\, h'(\bar{v})\, v_{\rho\rho\rho})\, v_0\, dt \\ &= v_{\rho\rho\rho}(1,\bar{\rho})\, v_0'(1). \end{aligned} \tag{6.8}$$

Differentiating (6.3) three times with respect to ρ shows that $(d^3T/d\rho^3)(\bar{\rho})=0$ if and only if $v_{\rho\rho\rho}(1,\bar{\rho})=0$. Thus it follows from (6.7) and (6.8) that $g_{xxx}(0,0)=0$ if and only if $(d^3T/d\rho^3)(\bar{\rho})=0$.

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